Theorem: Suppose N is a positive integer that requires d digits when expressed in base 10. Then, the number of digits required to express N in base 2 is at most $\left\lceil \frac{d}{\log 2} \right\rceil$ where $\lceil\ \rceil$ is the ceiling function.

Proof: We have that $10^{d-1} \leq N < 10^d$, and if b is the number of binary digits when N is written in base 2, then also $2^{b-1} \leq N < 2^b$. Hence, $2^{b-1} < 10^d$ and then $\log 2^{b-1} < \log 10^d$ so that $(b-1)\log 2 < d \log 10$. Therefore, $b - 1 < \frac{d}{\log 2}$. However, $b - 1$ is an integer, so that $b - 1 \leq \left\lfloor \frac{d}{\log 2} \right\rfloor$. Furthermore, $\log 2$ is irrational, so that $\frac{d}{\log 2}$ cannot be an integer. This implies that $\left\lfloor \frac{d}{\log 2} \right\rfloor = \left\lceil \frac{d}{\log 2} \right\rceil - 1$. Thus, it follows that $b - 1 \leq \left\lceil \frac{d}{\log 2} \right\rceil - 1$ and finally that $b \leq \left\lceil \frac{d}{\log 2} \right\rceil$.

The proof above relies heavily on the fact that $\log 2$ is irrational. This can be proved in the following lemma:

Lemma: $\log 2$ is irrational.

Proof: For sake of contradiction, suppose that $\log 2$ is rational. Then, $\log 2 = \frac{m}{n}$ where m and n are positive integers. This means $10^{\frac{m}{n}} = 2$ or that $10^m = 2^n$ so that $2^m 5^m = 2^n$. The left side of this equation is divisible by 5 but the right side is not, which is impossible. This is the contradiction we needed, thereby proving that $\log 2$ is irrational.

Here are a couple of examples of the above theorem.

Examples: Suppose we have the number 12345 and we wish to know how many digits we'll need to express this number in binary. The theorem tells us that we'll need at most $\left\lceil \frac{5}{\log 2} \right\rceil = \left\lceil \frac{5}{0.3010\cdots} \right\rceil = \lceil 16.6096\cdots \rceil = 17$. In fact, $12345_{10} = 11000000111001_2$ so that 14 binary digits are required for this number. For another example, $98765_{10} = 11000000111001101_2$ so that 17 binary digits are required for this number.

Theorem: Suppose N is a positive integer that requires b digits when expressed in base 2. Then, the number of digits required to express N in base 10 is at most $\lceil b \cdot \log 2 \rceil$ where $\lceil\ \rceil$ is the ceiling function.

Proof: We have that $2^{b-1} \leq N < 2^b$, and if d is the number of denary digits when N is written in base 10, then also $10^{d-1} \leq N < 10^d$. Hence, $10^{d-1} < 2^b$ and then $\log 10^{d-1} < \log 2^b$ so that $(d-1) \cdot \log 10 < b \cdot \log 2$. Therefore, $d - 1 < b \cdot \log 2$. However, $d - 1$ is an integer, so that $d - 1 \leq \lfloor b \cdot \log 2 \rfloor$. Furthermore, $\log 2$ is irrational, so that $b \cdot \log 2$ cannot be an integer. This implies that $\lfloor b \cdot \log 2 \rfloor = \lceil b \cdot \log 2 \rceil - 1$. Thus, it follows that $d - 1 \leq \lceil b \cdot \log 2 \rceil$ and finally that $d \leq \lceil b \cdot \log 2 \rceil$.

Examples: Suppose we have the binary number 1100011 and we wish to know how many digits we'll need to express this number in denary. The theorem tells us that we'll need at most $\lceil 7 \cdot \log 2 \rceil = \lceil 7 \cdot 0.3010 \cdots \rceil = \lceil 2.1072 \cdots \rceil = 3$ denary digits. In fact, $1100011_2 = 99_{10}$ so that 2 denary digits are required for this number. For another example, $1100100_2 = 100_{10}$ so that 3 denary digits are required for this number.